

# RETURN- AND HITTING-TIME DISTRIBUTIONS OF SMALL SETS IN INFINITE MEASURE PRESERVING SYSTEMS

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**ABSTRACT.** We study convergence of return- and hitting-time distributions of small sets in recurrent dynamical systems preserving an infinite measure  $\mu$ . In the presence of a Darling-Kac set with regularly varying wandering rate there is a scaling function suitable for all its subsets. In this case, we show that return distributions for a sequence  $(E_k)$  of sets with  $\mu(E_k) \rightarrow 0$  converge iff the corresponding hitting time distributions do, and we derive an explicit relation between the two limit laws. Some consequences of this result are discussed.

## 1. INTRODUCTION

The asymptotic behaviour of return- and hitting-time distributions of (very) small sets in ergodic probability preserving dynamical systems has been studied in great detail, and there is now a well-developed theory, both for specific types of maps and sets, and for general abstract systems.

For infinite measure preserving situations, however, results are scarce. Only recently some concrete classes of prototypical systems have been studied in [PS1], [PS2], and [PSZ2], where distributional limit theorems for certain natural sequences of sets were established. The purpose of the present note is to discuss some basic aspects of return- and hitting-time limits for asymptotically rare events in the setup of abstract infinite ergodic theory.

**General setup.** Throughout, *all measures are understood to be  $\sigma$ -finite*. We study *measure preserving transformations*  $T$  (not necessarily invertible) on a measure space  $(X, \mathcal{A}, \mu)$ , i.e. measurable maps  $T : X \rightarrow X$  for which  $\mu \circ T^{-1} = \mu$ . Here  $T$  will be *ergodic* (i.e. for  $A \in \mathcal{A}$  with  $T^{-1}A = A$  we have  $0 \in \{\mu(A), \mu(A^c)\}$ ) and *conservative* (meaning that  $\mu(A) = 0$  for all wandering sets, that is,  $A \in \mathcal{A}$  with  $T^{-n}A$ ,  $n \geq 1$ , pairwise disjoint), and thus *recurrent* (in that  $A \subseteq \bigcup_{n \geq 1} T^{-n}A \bmod \mu$  for  $A \in \mathcal{A}$ ). Our emphasis will be on the *infinite measure case*,  $\mu(X) = \infty$ .

For  $T$  such a conservative ergodic measure preserving transformation (*c.e.m.p.t.*) on  $(X, \mathcal{A}, \mu)$ , and any  $Y \in \mathcal{A}$ ,  $\mu(Y) > 0$ , we define the *first entrance time* function of  $Y$ ,  $\varphi_Y : X \rightarrow \mathbb{N} \cup \{\infty\}$  by  $\varphi_Y(x) := \min\{n \geq 1 : T^n x \in Y\}$ ,  $x \in X$ , and let  $T_Y x := T^{\varphi_Y(x)} x$ ,  $x \in X$ . When restricted to  $Y$ ,  $\varphi_Y$  is called the *first return time* of  $Y$ , and  $\mu|_{Y \cap \mathcal{A}}$  is invariant under the *first return map*,  $T_Y$  restricted to  $Y$ . If  $\mu(Y) < \infty$ , it is natural to regard  $\varphi_Y$  as a random variable on the probability space  $(X, \mathcal{A}, \mu_Y)$ , where  $\mu_Y(E) := \mu(Y)^{-1} \mu(Y \cap E)$ . By Kac' formula,

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$\int \varphi_Y d\mu_Y = \mu(X)/\mu(Y)$ . It is well known (see [A0]) that, for suitable reference sets  $Y$ , the distribution of this variable reflects important features of the system  $(X, \mathcal{A}, \mu, T)$ .

**Return- and hitting-time distributions for small sets.** Rather than focusing on a particular set  $Y$ , the present article studies the behaviour of such distributions for sequences  $(E_k)$  of sets of *positive* (meaning strictly positive) finite measure with  $\mu(E_k) \rightarrow 0$ , that is, for sequences of *asymptotically rare events*. As return times to small sets will typically be very large, the functions  $\varphi_{E_k}$  need to be normalized, which will be done using a certain scaling function  $\gamma$ .

We will thus study the distributions of random variables of the form  $\gamma(\mu(E)) \varphi_E$  on  $(E, E \cap \mathcal{A}, \mu_E)$ , with  $\mu(E)$  small, and call this the *(normalized) return time distribution of  $E$* ,

$$\text{law}_{\mu_E}[\gamma(\mu(E)) \varphi_E]$$

where, for  $\psi : X \rightarrow X'$  any  $\mathcal{A}$ - $\mathcal{A}'$ -measurable map and  $\nu \ll \mu$  a probability on  $(X, \mathcal{A})$ , we write  $\text{law}_\nu[\psi] := \nu \circ \psi^{-1}$ . In fact, we can use any such  $\nu$  as an initial distribution, in which case we refer to

$$\text{law}_\nu[\gamma(\mu(E)) \varphi_E]$$

as the *(normalized) hitting time distribution of  $E$  (under  $\nu$ )*. This leads to two different ways of looking at the  $\varphi_{E_k}$  for a sequence  $(E_k)$  as above: *asymptotic return distributions* of  $(E_k)$  are limits, as  $k \rightarrow \infty$ , of  $(\text{law}_{\mu_{E_k}}[\gamma(\mu(E_k)) \varphi_{E_k}])_{k \geq 1}$ , while *asymptotic hitting distributions* are limits of  $(\text{law}_\nu[\gamma(\mu(E_k)) \varphi_{E_k}])_{k \geq 1}$  for some fixed  $\nu$ . (The latter limits do not depend on the choice of  $\nu$ , and we often take  $\nu = \mu_Y$  for some nice set  $Y$ .) Understanding the relation between these two types of limits will be a central theme of this article.

It will be convenient to regard the distributions above as measures on  $[0, \infty]$ . Accordingly, we let  $\mathcal{F} := \{F : [0, \infty) \rightarrow [0, 1], \text{ non-decreasing and right-continuous}\}$  be the set of sub-probability distribution functions on  $[0, \infty)$ . For  $F, F_n \in \mathcal{F}$  ( $n \geq 1$ ) we write  $F_n \Rightarrow F$  for *vague convergence*, i.e.  $F_n(t) \rightarrow F(t)$  at all continuity points of  $F$ . For efficiency, we shall also use  $F_n(t) \Longrightarrow F(t)$  to express the same thing. (This allows us to use explicit functions of  $t$ .) If  $\sup F(t) = 1$  this is the usual *weak convergence* of probability distribution functions on  $[0, \infty)$ .

**Pointwise dual ergodicity, DK-sets, and  $\mathcal{U}$ -uniform sets.** Some classes of well-behaved infinite measure preserving systems are characterized by the existence of distinguished *reference sets*  $Y$ ,  $0 < \mu(Y) < \infty$ , with special properties. Those are often defined in terms of the *transfer operator*  $\hat{T} : L_1(\mu) \rightarrow L_1(\mu)$ , with  $\int_X u \cdot (v \circ T) d\mu = \int_X \hat{T}u \cdot v d\mu$  for all  $u \in L_1(\mu)$  and  $v \in L_\infty(\mu)$ . The operator  $\hat{T}$  naturally extends to  $\{u : X \rightarrow [0, \infty) \text{ } \mathcal{A}\text{-measurable}\}$ . It is a linear Markov operator,  $\int_X \hat{T}u d\mu = \int_X u d\mu$  for  $u \geq 0$ . The m.p.t.  $T$  is conservative and ergodic if and only if  $\sum_{k \geq 0} \hat{T}^k u = \infty$  a.e. for all  $u \in L_1^+(\mu) := \{u \in L_1(\mu) : u \geq 0 \text{ and } \mu(u) > 0\}$  or (equivalently) all  $u \in \mathcal{D}(\mu) := \{u \in L_1(\mu) : u \geq 0, \mu(u) = 1\}$ . By invariance of  $\mu$  we have  $\hat{T}1_X = 1_X$ .

A c.e.m.p.t.  $T$  on the space  $(X, \mathcal{A}, \mu)$  is said to be *pointwise dual ergodic* (cf. [A0], [A2]) if there is some sequence  $(a_n)$  in  $(0, \infty)$  such that

$$(1.1) \quad \frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k u \longrightarrow \mu(u) \cdot 1_X \quad \begin{array}{l} \text{a.e. on } X \text{ as } n \rightarrow \infty, \text{ for every} \\ u \in L_1(\mu) \text{ with } \mu(u) \neq 0. \end{array}$$

In this case,  $(a_n)$  (unique up to asymptotic equivalence, with  $a_n \rightarrow \infty$ ) is called a *return sequence* of  $T$ . W.l.o.g. we will assume throughout that  $a_n = a_T(n)$  for some strictly increasing continuous  $a_T : [0, \infty) \rightarrow [0, \infty)$  with  $a_T(0) = 0$ . For convenience, we shall call any homeomorphism of  $[0, \infty)$  a *scaling function*. Note that in case  $\mu(X) = \infty$ , we always have  $a_T(s) = o(s)$  as  $s \rightarrow \infty$ . Letting  $b_T$  denote the inverse function of  $a_T$ , we thus see that  $s = o(b_T(s))$  as  $s \rightarrow \infty$ . For later use define another scaling function  $\gamma_T : [0, \infty) \rightarrow [0, \infty)$  via  $\gamma_T(0) := 0$  and

$$(1.2) \quad \gamma_T(s) := 1/b_T(1/s) \quad \text{for } s > 0.$$

By Egorov's theorem, the convergence in (1.1) is uniform on suitable sets (depending on  $u$ ) of arbitrarily large measure. It is useful to identify specific pairs  $(u, Y)$ , with  $u \in \mathcal{D}(\mu)$  and  $Y \in \mathcal{A}$ ,  $0 < \mu(Y) < \infty$ , such that

$$(1.3) \quad \left\| 1_Y \cdot \left( \frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k u - 1_X \right) \right\|_\infty \longrightarrow 0 \quad \text{as } n \rightarrow \infty,$$

in which case we shall refer to  $Y$  as a  *$u$ -uniform set* (compare [A0], [T4]). In [PSZ2] the notion of a  *$\mathcal{U}$ -uniform set*  $Y$  was introduced. This means that  $\mathcal{U} \subseteq \mathcal{D}(\mu)$  is a class of densities such that the  $L_\infty(\mu)$ -convergence asserted in (1.3) holds uniformly in  $u \in \mathcal{U}$ , that is,

$$(1.4) \quad \sum_{k=0}^{n-1} \widehat{T}^k u \sim a_n \quad \begin{array}{l} \text{as } n \rightarrow \infty, \text{ uniformly mod } \mu \text{ on } Y, \\ \text{and uniformly in } u \in \mathcal{U}. \end{array}$$

A set  $Y$  which is  $\mu(Y)^{-1} \cdot 1_Y$ -uniform is called a *Darling-Kac (DK) set*, cf. [A0], [A3]. The existence of a uniform set implies pointwise dual ergodicity (as in Proposition 3.7.5 of [A0]), and the  $a_n$  in (1.3) then form a return sequence.

Several basic classes of infinite measure preserving systems, including Markov shifts and other Markov maps with good distortion properties (see [A0], [A3], and [T3]), as well as various non-Markovian interval maps (see [Z2], [Z4]), are known to possess DK-sets. Section 6 of [PSZ2] shows that a set  $Y$  on which  $T$  induces a Gibbs-Markov map  $T_Y$  is always  $\mathcal{U}$ -uniform for a reasonably large family  $\mathcal{U}$ .

Finer probabilistic statements about pointwise dual ergodic systems usually require  $a_T$  to be *regularly varying* with index  $\alpha \in [0, 1]$  (written  $a_T \in \mathcal{R}_\alpha$ ), meaning that for every  $c > 0$ ,  $a_T(ct)/a_T(t) \rightarrow c^\alpha$  as  $t \rightarrow \infty$  (see [BGT]). The asymptotics of  $a_T$  is intimately related to the return distribution  $\text{law}_{\mu_Y}[\varphi_Y]$  of any of its uniform sets  $Y$ : Write  $q_n(Y) := \mu_Y(\varphi_Y > n)$ ,  $n \geq 1$ , for the *tail probabilities* of  $\varphi_Y$ , and define the *wandering rate*  $(w_N(Y))_{N \geq 1}$  of  $Y$  as the (scaled) sequence of partial sums  $w_N(Y) := \mu(Y) \sum_{n=0}^{N-1} q_n(Y)$ . Then Aaronson's asymptotic renewal equation shows that  $(w_N) \in \mathcal{R}_{1-\alpha}$  for some  $\alpha \in [0, 1]$  implies

$$a_n \sim \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \frac{n}{w_n(Y)} \quad \text{as } n \rightarrow \infty,$$

whence  $a_T \in \mathcal{R}_\alpha$  (see Propositions 3.8.6 and 3.8.7 of [A0]).

**The concrete limit theorems of [PS1], [PS2], and [PSZ2].** The results of [PS1] and [PS2], were the starting point for the present investigations of return- and hitting-time limits in null-recurrent situations. They apply to certain skew-products which are “barely recurrent” in that  $(w_N) \in \mathcal{R}_1$  (corresponding to  $\alpha = 0$  above). In that case, only a seriously distorted version of the return-time function can have a nontrivial limit, see the discussion in Section 3 below. For natural sequences  $(E_k)$  of sets, those variables were shown to converge to the law with distribution function  $G_0(t) := t/(1+t)$ ,  $t \geq 0$ .

The skew-product structure was exploited through the use of local limit theorems. That approach has been extended to some (classical probabilistic)  $\alpha \in [0, 1/2]$  situations in [PSZ1]. To go beyond skew-products and the local limit technique, the notion of  $\mathcal{U}$ -uniform sets was introduced (and shown to work) in [PSZ2], which dealt with  $\alpha \in (0, 1]$  situations. For certain natural sequences  $(E_k)$ , suitably normalized return- (and hitting-) times  $\gamma(\mu(E_k)) \varphi_{E_k}$  were shown to converge to a law best expressed as the distribution of

$$(1.5) \quad \mathcal{H}_\alpha := \mathcal{E}^{\frac{1}{\alpha}} \mathcal{G}_\alpha, \quad \alpha \in (0, 1],$$

where  $\mathcal{E}$  and  $\mathcal{G}_\alpha$  are independent random variables, with  $\mathcal{E}$  exponentially distributed ( $\Pr[\mathcal{E} > t] = e^{-t}$  for  $t \geq 0$ ) and  $\mathcal{G}_\alpha$ ,  $\alpha \in (0, 1)$ , following the one-sided stable law of order  $\alpha$  ( $\mathbb{E}[\exp(-s\mathcal{G}_\alpha)] = \exp(-s^\alpha)$  for  $s \geq 0$ ), while  $\mathcal{G}_1 = 1$ . We use  $H_\alpha(t) := \Pr[\mathcal{H}_\alpha \leq t]$ ,  $t \geq 0$ , to denote the distribution function of  $\mathcal{H}_\alpha$ .

**Outline of results.** In contrast to references [PS1], [PS2], [PSZ2] mentioned before, which study specific classes of systems and particular types of sequences  $(E_k)$ , the present note discusses the asymptotics of general asymptotically rare sequence  $(E_k)$  in an abstract setup.

We first discuss the basic question of how to normalize the functions  $\varphi_E$ , and show that it is impossible to find a scaling function  $\gamma$  such that  $\gamma(\mu(E))$  captures the order of magnitude of  $\varphi_E$  for all (small) sets  $E$ . However, if  $T$  admits a DK-set  $Y$  with regularly varying return sequence, then there is some  $\gamma = \gamma_T$  which works for every  $E$  contained in  $Y$ .

In this very setup, we then prove that, for every asymptotically rare sequence  $(E_k)$  inside  $Y$ , the return-time distributions converge iff the hitting-time distributions converge. We also clarify the relation between the respective limit laws. The latter allows us to characterize convergence to the specific limit laws which occurred in [PS1], [PS2], and [PSZ2] as asymptotic equivalence of return- and hitting-time distributions. This gives an alternative approach to the limit theorem of [PSZ2].

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## 2. HOW TO NORMALIZE RETURN-TIMES OF SMALL SETS

We collect some facts regarding the order of magnitude of a return-time variable  $\varphi_E$ , focusing on its relation to the measure of the set  $E$ . We first record some basic observations to point out some of the difficulties which are inevitable when dealing with infinite measures. We then formulate the main results of this section.

**Scaling return-times in finite measure systems.** As a warm-up, assume first that  $(X, \mathcal{A}, \mu, T)$  is ergodic and measure preserving, with  $\mu(X) < \infty$ . Kac' formula  $\int_E \varphi_E d\mu_E = \mu(X)/\mu(E)$  for the expectation of the return-time of an arbitrary set  $E \in \mathcal{A}$  with  $\mu(E) > 0$  not only shows that  $\mu(E) \varphi_E$  is the canonical choice if we wish to use *normalized return times*, but also yields the simple estimate  $\mu_E(\mu(E) \varphi_E > t) \leq 1/t, t > 0$ . The latter can be read as an explicit version of the trivial statement that the family of all normalized return distributions,

$$(2.1) \quad \{\text{law}_{\mu_E}[\mu(E) \varphi_E] : E \in \mathcal{A}, \mu(E) > 0\}, \text{ is tight.}$$

We record an obvious consequence of this by also stating that for every  $\eta > 0$ ,

$$(2.2) \quad \{\text{law}_{\mu_E}[\varphi_E] : E \in \mathcal{A}, \mu(E) \geq \eta\} \text{ is tight.}$$

The relevance of these trivialities for the present paper lies in the fact that they break down when  $\mu(X) = \infty$ .

**Scaling return-times in infinite measure systems - difficulties.** Now let  $(X, \mathcal{A}, \mu, T)$  be a c.e.m.p.t. system with  $\mu(X) = \infty$ . We are interested in the return distributions of sets of positive finite measure. Kac' formula remains valid in that  $\int_E \varphi_E d\mu_E = \infty$  for every set  $E \in \mathcal{A}$  with  $\mu(E) > 0$ , but it no longer provides us with a canonical normalization for  $\varphi_E$ . Indeed, the situation is more complicated than in the finite measure regime:

**Proposition 2.1 (Basic (non-)tightness properties of return distributions).**

*Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$  with  $\mu(X) = \infty$ .*

*a) The family of return distributions of large sets  $E$  is not tight: For every  $\eta > 0$ ,*

$$(2.3) \quad \{\text{law}_{\mu_E}[\varphi_E] : E \in \mathcal{A}, \mu(E) \geq \eta\} \text{ is not tight.}$$

*b) Locally, the family of return distributions of large sets  $E$  is tight: Let  $Y \in \mathcal{A}$  with  $0 < \mu(Y) < \infty$ . Then for every  $\eta > 0$ ,*

$$(2.4) \quad \{\text{law}_{\mu_E}[\varphi_E] : E \in Y \cap \mathcal{A}, \mu(E) \geq \eta\} \text{ is tight.}$$

*c) Even locally, the family of return distributions of arbitrary sets  $E$  with normalization  $\mu(E)$  is not tight: Let  $Y \in \mathcal{A}$  with  $0 < \mu(Y) < \infty$ . Then*

$$(2.5) \quad \{\text{law}_{\mu_E}[\mu(E) \varphi_E] : E \in Y \cap \mathcal{A}, \mu(E) > 0\} \text{ is not tight.}$$

Statement c) of the proposition shows that  $\mu(E)$  is not an appropriate normalizing factor. In Theorem 2.2 below we identify, under additional assumptions, a scaling function  $\gamma : [0, \infty) \rightarrow [0, \infty)$  for which  $\gamma(\mu(E))$  gives a suitable normalization, at least locally, that is, inside certain reference sets  $Y$ . To appreciate this, observe first that there cannot be a global statement of this type. In fact, there never is a scaling function  $\gamma$  which works locally inside every set  $Y$  of positive finite measure:

**Theorem 2.1 (No universal scale for return times).** *Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$  with  $\mu(X) = \infty$ , and let  $\gamma : [0, \infty) \rightarrow [0, \infty)$  be a scaling function. Then there is some  $Y \in \mathcal{A}$  with  $0 < \mu(Y) < \infty$  such that*

$$(2.6) \quad \{\text{law}_{\mu_E}[\gamma(\mu(E)) \varphi_E] : E \in Y \cap \mathcal{A}, \mu(E) > 0\} \text{ is not tight.}$$

**Scaling return-times in infinite measure systems - a positive result.**

Nonetheless, there are systems which possess distinguished reference sets  $Y$  of positive finite measure inside which all sets comply with an explicit common scale function. The main positive result of the present section is

**Theorem 2.2 (Tightness of normalized return distributions inside DK-sets).**

Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$ ,  $\mu(X) = \infty$ . Assume that  $Y \in \mathcal{A}$  is a DK-set and that  $a_T \in \mathcal{R}_\alpha$  for some  $\alpha \in (0, 1]$ . Let  $b_T$  be the inverse function of  $a_T$ , and

$$(2.7) \quad \gamma_T(s) := 1/b_T(1/s), \quad s > 0.$$

Then  $\gamma_T \in \mathcal{R}_{1/\alpha}(0^+)$  and the family

$$(2.8) \quad \{\text{law}_{\mu_E}[\gamma_T(\mu(E)) \varphi_E] : E \in Y \cap \mathcal{A}, \mu(E) > 0\} \text{ is tight.}$$

The situation inside a DK-set with regularly varying return sequence therefore is not as wild as it is for arbitrary sets. The limit theorems of [PSZ1], and [PSZ2] show that for various interesting and natural sequences  $(E_k)$  in  $Y$  with  $\mu(E_k) \rightarrow 0$ , the normalized return times  $\gamma_T(\mu(E_k)) \varphi_{E_k}$  do have non-trivial continuous limit distributions concentrated on  $(0, \infty)$ . (We review and extend this in Section 5 below.) For those sequences,  $(\gamma_T(\mu(E_k)))$  captures the order of the  $\varphi_{E_k}$  exactly. Theorem 2.2 confirms that  $\gamma_T(\mu(E))$  gives a tight scaling for *all* subsets  $E \in Y \cap \mathcal{A}$  of positive measure. We conclude with the very easy observation that there are always sets with exceptionally short returns, which elude any given scale function.

**Proposition 2.2 (Sets with very short returns).** Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$ , and  $\gamma : [0, \infty) \rightarrow [0, \infty)$  a scaling function. Assume that  $Y \in \mathcal{A}$  satisfies  $\mu(Y) > 0$ , then there are sets  $E_k \in Y \cap \mathcal{A}$ ,  $k \geq 1$ , such that  $0 < \mu(E_k) \rightarrow 0$  and

$$(2.9) \quad \mu_{E_k}(\gamma(\mu(E_k)) \varphi_{E_k} \leq t) \rightarrow 1 \quad \text{for } t \geq 1.$$

**Proofs for this section.** The proof of Theorem 2.2 is deferred to Section 4, since it will use a result from Section 3. Here, we begin with the

**Proof of Proposition 2.1. a)** We show that there are  $E_k \in \mathcal{A}$  with  $\mu(E_k) = \eta$  such that  $\varphi_{E_k} \geq k$  on  $E_k$  for  $k \geq 1$ .

Note first that an infinite measure space allowing a c.e.m.p. map  $T$  is necessarily nonatomic. Take some  $Y \in \mathcal{A}$  with  $0 < \mu(Y) < \infty$ , and set  $Y_0 := Y$  and  $Y_n := Y^c \cap \{\varphi_Y = n\}$ ,  $n \geq 1$ . For any  $k \geq 1$ , the set  $A_k := \bigcup_{j \geq 1} Y_{jk}$  satisfies  $\varphi_{A_k} \mid_{A_k} \geq k$  and therefore  $\varphi_E \mid_E \geq k$  holds for all  $E \in A_k \cap \mathcal{A}$ . Since  $\mu(A_k) = \infty$  and  $\mu$  is nonatomic,  $A_k$  has a subset  $E_k$  with  $\mu(E_k) = \eta$ .

**b)** We first prove that for every  $E \in Y \cap \mathcal{A}$  with  $\mu(E) > 0$ , and any  $m, n \geq 1$ ,

$$(2.10) \quad \mu_E(\varphi_E > mn) \leq \frac{\mu(Y)}{\mu(E)} \left( \frac{1}{m} + m \mu_Y(\varphi_Y > n) \right).$$

Note first that decomposing an excursion from  $E$  into consecutive excursions from  $Y$ , we can represent  $\varphi_E$  as

$$(2.11) \quad \varphi_E = \sum_{j=0}^{\varphi_E^Y - 1} \varphi_Y \circ T_Y^j \quad \text{on } Y,$$

where  $\varphi_E^Y(x) := \inf\{i \geq 1 : T_Y^i x \in E\}$  denotes the first entrance time of  $E$  under the induced map  $T_Y$ . This reveals that

$$(2.12) \quad E \cap \{\varphi_E > mn\} \subseteq (E \cap \{\varphi_E^Y > m\}) \cup \left( Y \cap \bigcup_{j=0}^{m-1} T_Y^{-j} \{\varphi_Y > n\} \right).$$

Applying Kac' formula to  $T_Y$  gives  $\mu_E(\varphi_E^Y > m) \leq \mu(Y)/(m\mu(E))$ , and since  $T_Y$  preserves  $\mu_Y$ , it is clear that  $\mu_Y(\bigcup_{j=0}^{m-1} T_Y^{-j} \{\varphi_Y > n\}) \leq m\mu_Y(\varphi_Y > n)$ . Combining these yields (2.10).

Now take any  $\varepsilon > 0$ . First choose  $m \geq 1$  so large that  $\frac{\mu(Y)}{\eta m} < \frac{\varepsilon}{2}$ , then pick  $n \geq 1$  so large that  $\frac{\mu(Y)}{\eta} m\mu_Y(\varphi_Y > n) < \frac{\varepsilon}{2}$  as well. Now (2.10) shows that  $\mu_E(\varphi_E > mn) < \varepsilon$  whenever  $E \in Y \cap \mathcal{A}$  satisfies  $\mu(E) \geq \eta$ .

c) We prove that there are sets  $E_k \in Y \cap \mathcal{A}$  with  $0 < \mu(E_k) < 1/k$  and

$$(2.13) \quad \mu_{E_k}(\mu(E_k)\varphi_{E_k} > k) > (k-1)/k \quad \text{for } k \geq 1.$$

We can assume w.l.o.g. that  $\mu(Y) = 1$ . Fix any  $k \geq 1$ . Since, according to Kac' formula,  $\int_Y \varphi_Y d\mu = \infty$ , we have  $m^{-1} \sum_{j=0}^{m-1} \varphi_Y \circ T_Y^j \rightarrow \infty$  a.e. on  $Y$  by (an obvious extension of) the ergodic theorem. An Egorov-type argument shows that there is some  $Z \in Y \cap \mathcal{A}$  with  $\mu(Y \setminus Z) < 1/(2k)$  and an integer  $M > k$  such that

$$(2.14) \quad \sum_{j=0}^{m-1} \varphi_Y \circ T_Y^j > 2mk \quad \text{on } Z \text{ for } m \geq M.$$

Recalling the representation (2.11), we conclude that for every  $E \in Y \cap \mathcal{A}$ ,

$$(2.15) \quad \mu(E)\varphi_E > 2M\mu(E)k \quad \text{on } Z \cap \{\varphi_E^Y \geq M\}.$$

Now appeal to the Rokhlin lemma to obtain some  $F \in Y \cap \mathcal{A}$  for which the sets  $F_i := T_Y^{-i}F$ ,  $i \in \{0, \dots, M-1\}$  are pairwise disjoint, and  $\mu(Y \setminus \bigcup_{i=0}^{M-1} F_i) < 1/(M+1)$ . In particular,  $1/(M+1) < \mu(F_i) \leq 1/M$  for each  $i$ . Observing that  $\mu(\bigcup_{i=0}^{M-1} F_i \setminus Z) < \mu(\bigcup_{i=0}^{M-1} F_i)/k$ , we see that there is some  $i_0 \in \{0, \dots, M-1\}$  for which

$$(2.16) \quad \mu_{F_{i_0}}(F_{i_0} \setminus Z) < 1/k.$$

Let  $E_k := F_{i_0}$ , then  $E_k \in Y \cap \mathcal{A}$  with  $1/(M+1) < \mu(E_k) < 1/k$ . On the other hand, by the Rokhlin tower structure, we have  $\varphi_{E_k}^Y \geq M$  on  $E_k$ , and can thus employ the estimate (2.15) to see that

$$(2.17) \quad \mu(E_k)\varphi_{E_k} > k \quad \text{on } Z \cap E_k.$$

In view of (2.16), this implies our claim (2.13).  $\square$

**Proof of Theorem 2.1.** We are going to prove that there are  $E_k \in \mathcal{A}$  with  $\mu(E_k) \searrow 0$ , for which  $\gamma(\mu(E_k))\varphi_{E_k} \geq k$  on  $E_k$  for  $k \geq 1$ . To then obtain a set  $Y$  as promised in the statement of the theorem, pick a subsequence  $(E_{k_j})_{j \geq 1}$  such that  $\sum_{j \geq 1} \mu(E_{k_j}) < \infty$ , and define  $Y := \bigcup_{j \geq 1} E_{k_j}$ .

Starting from  $n_1 := 1$  we first take some strictly increasing sequence  $(n_l)_{l \geq 1}$  of integers which satisfies

$$(2.18) \quad n_{l+1} > (l+1) \max \left( n_l, 1/\gamma \left( \frac{1}{l+1} \right) \right) \quad \text{for } l \geq 1.$$

Next, construct a sequence  $(a_n)_{n \geq 1}$  in  $(0, \infty)$  by setting  $a_{n_1} := a_{n_2} := 1$  and  $a_{n_{l+1}} := 2n_{l+1}/l$  for  $l \geq 2$ , and by requiring that  $(a_n)$  be constant on each  $\{n_l, \dots, n_{l+1} - 1\}$ . Then,  $1 \geq a_n/n \rightarrow 0$  as  $n \rightarrow \infty$ .

By Proposition 3.8.2 of [A0], there is some  $Z \in \mathcal{A}$ ,  $0 < \mu(Z) \leq 1$ , such that

$$(2.19) \quad w_n(Z) \geq a_n \quad \text{for } n \geq 1.$$

(The fact that one can ensure  $\mu(Z) \leq \sup_n a_n/n = 1$  is made explicit in the proof of that result.) Let  $Z_0 := Z$  and  $Z_n := Z^c \cap \{\varphi_Z = n\}$ ,  $n \geq 1$ , then  $z_n := \mu(Z_n) \searrow 0$  and  $w_n(Z) = \sum_{k=0}^{n-1} z_k$ . In particular, (2.19) guarantees that

$$(2.20) \quad \frac{1}{n_{l+1}} \sum_{k=0}^{n_{l+1}-1} z_k \geq 2/l \quad \text{for } l \geq 2.$$

But then,

$$(2.21) \quad \begin{aligned} z_{n_l} &\geq \frac{1}{n_{l+1} - n_l} \sum_{k=n_l}^{n_{l+1}-1} z_k \geq \frac{1}{n_{l+1} - n_l} \left( \sum_{k=0}^{n_{l+1}-1} z_k - n_l \right) \\ &\geq \frac{1}{n_{l+1}} \sum_{k=0}^{n_{l+1}-1} z_k - \frac{n_l}{n_{l+1} - n_l} > \frac{2}{l} - \frac{1}{l} = \frac{1}{l} \quad \text{for } l \geq 2, \end{aligned}$$

since  $n_{l+1} > (l+1)n_l$  by construction, see (2.18).

As  $\gamma$  is non-decreasing, (2.21) shows that  $\gamma(\mu(Z_{n_l})) \geq \gamma(1/l)$  for such  $l$ . Moreover, since the  $Z_n$  are pairwise disjoint, and  $TZ_{n+1} \subseteq Z_n$  for  $n \geq 0$ , it is clear that  $\varphi_{Z_n} > n$  on  $Z_n$ . We thus find (using the second lower bound from (2.18)) that

$$\gamma(\mu(Z_{n_l})) \varphi_{Z_{n_l}} \geq \gamma(1/l) n_l > l \quad \text{on } Z_{n_l} \quad \text{for } l \geq 2.$$

Now take  $E_l := Z_{n_l}$ ,  $l \geq 2$ , and  $E_1 := E_2$ . □

**Proof of Proposition 2.2.** Assume w.l.o.g. that  $\mu(Y) = 1$ . We construct  $E_k \in Y \cap \mathcal{A}$ ,  $k \geq 1$ , s.t.  $\mu(E_k) = 1/k$  and  $\mu_{E_k}(\varphi_{E_k} > 1) \leq 1/k$ . Then  $(E_k)$  satisfies (2.9).

Fix any  $k \geq 1$ . By the Rokhlin lemma, there is some  $F \in Y \cap \mathcal{A}$ ,  $\mu(F) > 0$ , such that the sets  $F_i := T_Y^{-i} F$ ,  $i \in \{0, \dots, k-1\}$  are pairwise disjoint. Since  $\mu$  is nonatomic, there is some  $F' \in F \cap \mathcal{A}$  such that  $\mu(F') = 1/k^2$ . Set  $E_k := \bigcup_{i=0}^{k-1} T_Y^{-i} F'$  (disjoint), then  $\mu(E_k) = 1/k$ , and  $\varphi_{E_k} = 1$  on  $E_k \setminus F'$ . □

### 3. RETURN-TIME LIMITS VERSUS HITTING-TIME LIMITS

This section presents the main results of the present paper. Our proofs of several other results (including Theorem 2.2 above) rely on this theorem.

We clarify the relation between asymptotic return-time distributions and asymptotic hitting-time distributions in the situation of Theorem 2.2, where a suitable scaling function has been identified.

**Hitting versus returning in finite measure systems.** Recall that in the probability-preserving setup, a simple general principle relates limit laws for normalized hitting-times and for normalized return-times to each other (see [HLV]):



**Theorem HLV (Return and hitting-time limits for finite measure).** *Let  $T$  be an ergodic m.p.t. on  $(X, \mathcal{A}, \mu)$ ,  $\mu(X) = 1$ . Suppose that  $E_k$ ,  $k \geq 1$ , are sets of positive measure with  $\mu(E_k) \rightarrow 0$ .*

*Then the normalized return-time distributions of the  $E_k$  converge in that*

$$(3.1) \quad \mu_{E_k}(\mu(E_k) \varphi_{E_k} \leq t) \implies \tilde{F}(t) \quad \text{as } k \rightarrow \infty$$

*for some  $\tilde{F} \in F$ , if and only if the normalized hitting-time distributions converge,*

$$(3.2) \quad \mu_Y(\mu(E_k) \varphi_{E_k} \leq t) \implies F(t) \quad \text{as } k \rightarrow \infty$$

*for some  $F \in F$ . In this case the limit laws satisfy*

$$(3.3) \quad F(t) = \int_0^t [1 - \tilde{F}(s)] ds \quad \text{for } t \geq 0.$$

If (3.3) is satisfied,  $F$  is sometimes called the *integrated tail distribution* of  $\tilde{F}$ .

**Hitting versus returning in infinite measure systems.** We address the obvious question of how the two different types of limit theorems are related in null-recurrent situations, and prove an abstract result in the spirit of Theorem HLV which applies to infinite measure preserving maps possessing a Darling-Kac set with regularly varying return sequence. We use the normalization discussed in the preceding section. The following result confirms once again that the latter is a sensible choice.

**Theorem 3.1 (Return- versus hitting-time limits inside DK-sets).** *Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$ ,  $\mu(X) = \infty$ . Assume that  $Y \in \mathcal{A}$  is a DK-set and that  $a_T \in \mathcal{R}_\alpha$  for some  $\alpha \in (0, 1]$ . Define  $\gamma_T$  by  $\gamma_T(s) := 1/b_T(1/s)$ ,  $s > 0$ .*

*Suppose that  $E_k \subseteq Y$ ,  $k \geq 1$ , are sets of positive measure with  $\mu(E_k) \rightarrow 0$ . Then the normalized return-time distributions of the  $E_k$  converge in that*

$$(3.4) \quad \mu_{E_k}(\gamma_T(\mu(E_k)) \varphi_{E_k} \leq t) \implies \tilde{F}(t) \quad \text{as } k \rightarrow \infty$$

*for some  $\tilde{F} \in \mathcal{F}$ , if and only if the normalized hitting-time distributions converge,*

$$(3.5) \quad \mu_Y(\gamma_T(\mu(E_k)) \varphi_{E_k} \leq t) \implies F(t) \quad \text{as } k \rightarrow \infty$$

*for some  $F \in \mathcal{F}$ . In this case the limit laws satisfy*

$$(3.6) \quad F(t) = \int_0^t [1 - \tilde{F}(s)] \alpha (t-s)^{\alpha-1} ds \quad \text{for } t \geq 0.$$

**Remark 3.1 (Some comments and consequences).** We record the following:

- a) In the  $\alpha = 1$  case of a “barely infinite” measure, (3.6) reduces to (3.3).
- b) In (3.6), the function  $\tilde{F}$  clearly determines  $F$ , and vice versa.
- c) Relation (3.6) shows that  $F$  is necessarily continuous on  $[0, \infty)$  with  $F(0) = 0$ . Moreover, since  $\int_0^t \alpha (t-s)^{\alpha-1} ds = t^\alpha \rightarrow \infty$  as  $t \rightarrow \infty$ , it is immediate that  $\tilde{F}(s) \rightarrow 1$  as  $s \rightarrow \infty$ , so that  $\tilde{F}$  is a *probability* distribution function on  $[0, \infty)$ .
- d) By the previous remark and Theorem 2.1 we see that under the assumptions of Theorem 3.1 there are always sets  $Y' \in \mathcal{A}$ ,  $0 < \mu(Y') < \infty$ , inside which the conclusion of Theorem 3.1 fails.
- e) If  $(E_k)$  is a sequence such that (3.4) takes place with  $\tilde{F} = 1_{[0, \infty)}$ , that is  $\text{law}_{\mu_{E_k}}[\gamma_T(\mu(E_k)) \varphi_{E_k}] \implies \delta_0$  (by Proposition 2.2 the set  $Y$  always contains such

a sequence), then (3.5) holds with  $F = 0$ , so that  $\text{law}_{\mu_Y}[\gamma_T(\mu(E_k)) \varphi_{E_k}] \Rightarrow \delta_\infty$ .  
**f)** If for some  $\alpha \in (0, 1]$  the sub-distribution function  $F \in \mathcal{F}$  on  $[0, \infty)$  satisfies

$$(3.7) \quad F(t) = \int_0^t [1 - F(s)] \alpha (t - s)^{\alpha-1} ds \quad \text{for } t \geq 0,$$

then  $F = H_\alpha$ , the distribution function of the random variable  $\mathcal{H}_\alpha = \mathcal{E}^{\frac{1}{\alpha}} \mathcal{G}_\alpha$ . This is contained in Lemma 7 of [PSZ1]. (Note that the parameter  $\alpha$  appearing in that Lemma is not the same as our  $\alpha$ . In the notation of the present paper it equals  $1/(1 - \alpha)$ ).

**g)** In (3.5) the measure  $\mu_Y$  can be replaced by any fixed probability measure  $\nu \ll \mu$ . (Convergence of hitting-time distributions is always a case of *strong distributional convergence* in the sense of [A0], see Corollary 5 of [Z7].)

Theorem 3.1 is a consequence of the following result where return- and hitting times are distorted by the nonlinear function  $a_T$ . Theorem 3.2 is more general in that it also gives nontrivial information about the “barely recurrent”  $\alpha = 0$  case, which is excluded in Theorem 3.1. This case is of interest, because of very natural examples (recurrent random walks on  $\mathbb{Z}^2$  and recurrent  $\mathbb{Z}^2$ -extensions including the Lorentz process, see [PS1] and [PS2], and slowly recurrent random walks on  $\mathbb{R}$ , [PSZ1]) in which (for typical sequences  $(E_k)$ ) the  $\mu(E_k) a_T(\varphi_{E_k})$  have been shown to converge to the limit law with distribution function  $G_0(t) := t/(1 + t)$ ,  $t \geq 1$ .

**Theorem 3.2 (Distorted return- versus hitting-time limits in DK-sets).**  
*Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$ ,  $\mu(X) = \infty$ . Assume that  $Y \in \mathcal{A}$  is a DK-set and that  $a_T \in \mathcal{R}_\alpha$  for some  $\alpha \in [0, 1]$ .*

*Suppose that  $E_k \subseteq Y$ ,  $k \geq 1$ , are sets of positive measure with  $\mu(E_k) \rightarrow 0$ . Then the distorted return-time distributions of the  $E_k$  converge,*

$$(3.8) \quad \mu_{E_k}(\mu(E_k) a_T(\varphi_{E_k}) \leq t) \Rightarrow \tilde{G}(t) \quad \text{as } k \rightarrow \infty$$

*for some  $\tilde{G} \in \mathcal{F}$ , if and only if the distorted hitting-time distributions converge,*

$$(3.9) \quad \mu_Y(\mu(E_k) a_T(\varphi_{E_k}) \leq t) \Rightarrow G(t) \quad \text{as } k \rightarrow \infty$$

*for some  $G \in \mathcal{F}$ . In this case the limit laws satisfy, for  $t \geq 0$ ,*

$$(3.10) \quad G(t) = \begin{cases} t \int_0^1 [1 - \tilde{G}(t(1-r)^\alpha)] \alpha r^{\alpha-1} dr & \text{if } \alpha \in (0, 1], \\ t[1 - \tilde{G}(t)] & \text{if } \alpha = 0, \end{cases}$$

*and  $G$  is continuous on  $[0, \infty)$ .*

The proofs of the two theorems are given in the next section. In fact, for  $\alpha \in (0, 1]$ , the statement of Theorem 3.2 is equivalent to Theorem 3.1, with  $(\tilde{G}(t), G(t)) = (\tilde{F}(t^{1/\alpha}), F(t^{1/\alpha}))$ , see Lemma 4.2 below. Remark 3.1 translates accordingly. Regarding the  $\alpha = 0$  case, some related facts are recorded in

**Remark 3.2 (More comments).** **a)** If  $G \in \mathcal{F}$  satisfies

$$(3.11) \quad G(t) = t(1 - G(t)) \quad \text{for } t > 0,$$

then  $G(t) = G_0(t) = t/(1+t)$ , the  $\alpha = 0$  limit law from [PS1], [PS2], and [PSZ1].

**b)** Regarding the right-hand side of (3.10), note that for every  $\tilde{G} \in \mathcal{F}$  and every continuity point  $t > 0$  of  $\tilde{G}$  we have

$$\int_0^1 [1 - \tilde{G}(t(1-r)^\alpha)] \alpha r^{\alpha-1} dr \longrightarrow 1 - \tilde{G}(t) \quad \text{as } \alpha \searrow 0.$$

While the  $a_T(\varphi_{E_k})$  do exhibit nontrivial asymptotic distributional behaviour in the interesting  $\alpha = 0$  situations mentioned above, the original  $\varphi_{E_k}$  do not. A formal version of this statement is immediate from the following fact.

**Proposition 3.1 (No way back from  $\ell(R_n)$  to  $R_n$ ).** *Let  $\ell : [0, \infty) \rightarrow [0, \infty)$  be a slowly varying homeomorphism. Assume that  $(R_n)$  is a sequence of  $[0, \infty]$ -valued random variables with  $R_n \Rightarrow \infty$  for which  $(\ell(R_n))$  has a continuous limit distribution on  $(0, \infty)$ , that is, there are  $(\gamma_n)$  in  $(0, \infty)$  and a continuous random variable  $L$  with  $0 < L < \infty$  a.s., such that*

$$(3.12) \quad \gamma_n \ell(R_n) \Rightarrow L \quad \text{as } n \rightarrow \infty.$$

*Then any limit distribution of  $(R_n)$  is concentrated on  $\{0, \infty\}$ , that is, if*

$$(3.13) \quad \eta_n R_n \Rightarrow R \quad \text{as } n \rightarrow \infty,$$

*for some  $(\eta_n)$  in  $(0, \infty)$  and some random variable  $R$ , then  $\Pr[R \in \{0, \infty\}] = 1$ .*

*Proof.* Assume, for a contradiction, that (3.13) holds and  $\Pr[e^{-c} < R \leq e^c] > 0$  for some  $c > 1$ . Then there are  $\kappa > 0$  and  $n_0 \geq 1$  such that the events  $A_n := \{e^{-2c} < \eta_n R_n \leq e^{2c}\}$  satisfy  $\Pr[A_n] \geq \kappa$  for  $n \geq n_0$ . Set  $\tilde{\gamma}_n := \ell(1/\eta_n)^{-1}$  and  $t_n^\pm := \tilde{\gamma}_n \ell(e^{\pm 2c}/\eta_n)$ , so that the above becomes

$$(3.14) \quad \Pr[t_n^- < \tilde{\gamma}_n \ell(R_n) \leq t_n^+] \geq \kappa \quad \text{for } n \geq n_0.$$

Since  $R_n \Rightarrow \infty$ , we have  $\eta_n \rightarrow 0$ , and slow variation of  $\ell$  yields  $t_n^\pm \rightarrow 1$ .

Passing to a subsequence if necessary, we may assume that there is some  $[0, \infty]$ -valued random variable  $\tilde{L}$  such that

$$(3.15) \quad \tilde{\gamma}_n \ell(R_n) \Rightarrow \tilde{L} \quad \text{as } n \rightarrow \infty.$$

Now (3.14) and  $t_n^\pm \rightarrow 1$  clearly imply  $\Pr[\tilde{L} = 1] \geq \kappa > 0$ . In view of the standard *convergence of types* theorem, however, this contradicts our assumption (3.12), where  $L$  is a continuous variable in  $(0, \infty)$ .  $\square$

**Robustness of limiting behaviour.** As a first application of the theorem, we show that the asymptotic behaviour of both return- and hitting-time distributions for an asymptotically rare sequence  $(E_k)$  is robust under small modifications of the sets.

**Proposition 3.2 (Robustness of return- and hitting time limits).** *Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$ ,  $\mu(X) = \infty$ . Assume that  $Y \in \mathcal{A}$  is a DK-set and that  $a_T \in \mathcal{R}_\alpha$  for some  $\alpha \in (0, 1]$ . Suppose that  $E_k, E'_k \subseteq Y$ ,  $k \geq 1$ , are sets of positive measure with  $\mu(E_k) \rightarrow 0$ .*

*If  $\mu(E_k \triangle E'_k) = o(\mu(E_k))$  as  $k \rightarrow \infty$ , then, for arbitrary  $F, \tilde{F} \in \mathcal{F}$ ,*

$$(3.16) \quad \mu_{E_k}(\gamma_T(\mu(E_k)) \varphi_{E_k} \leq t) \Rightarrow \tilde{F}(t) \quad \text{iff} \quad \mu_{E'_k}(\gamma_T(\mu(E'_k)) \varphi_{E'_k} \leq t) \Rightarrow \tilde{F}(t)$$

and

$$(3.17) \quad \mu_Y(\gamma_T(\mu(E_k)) \varphi_{E_k} \leq t) \Rightarrow F(t) \quad \text{iff} \quad \mu_Y(\gamma_T(\mu(E'_k)) \varphi_{E'_k} \leq t) \Rightarrow F(t).$$

*Proof.* Abbreviating  $\gamma_T =: \gamma$  we first observe that for sets  $A, B \in \mathcal{A}$  with  $\mu(A) > 0$  the hitting time distributions under  $\mu_Y$ , encoded in  $F_A(t) := \mu_Y(\gamma(\mu(A)) \varphi_A \leq t)$ ,  $t \geq 0$ , satisfy

$$(3.18) \quad F_A \left( \frac{\gamma(\mu(A))}{\gamma(\mu(A \cup B))} t \right) \leq F_{A \cup B}(t) \leq F_A(t) + F_B \left( \frac{\gamma(\mu(B))}{\gamma(\mu(A))} t \right).$$

Indeed, since  $\gamma$  is non-decreasing, and since  $\varphi_{A \cup B} = \varphi_A \wedge \varphi_B$ , we have

$$\begin{aligned} F_{A \cup B}(t) &\leq \mu_Y(\gamma(\mu(A)) \varphi_{A \cup B} \leq t) \\ &\leq \mu_Y(\gamma(\mu(A)) \varphi_A \leq t) + \mu_Y(\gamma(\mu(A)) \varphi_B \leq t), \end{aligned}$$

which gives the upper estimate in (3.18). The lower estimate follows from the same two properties, as  $F_{A \cup B}(t) \geq \mu_Y(\gamma(\mu(A \cup B)) \varphi_A \leq t)$ .

Assume now that  $(A_k)$  and  $(B_k)$  are sequences in  $\mathcal{A}$ , satisfying  $\mu(A_k) > 0$  and

$$\mu(B_k) = o(\mu(A_k)) \quad \text{as } k \rightarrow \infty.$$

Monotonicity and regular variation of  $\gamma$  (with index  $\frac{1}{\alpha} \geq 1$ ) immediately entail

$$\gamma(\mu(B_k)) = o(\gamma(\mu(A_k))) \quad \text{and} \quad \gamma(\mu(A_k \cup B_k)) \sim \gamma(\mu(A_k))$$

as  $k \rightarrow \infty$ . In view of this, (3.18) is easily seen to imply that

$$F_{A_k} \implies F \quad \text{iff} \quad F_{A_k \cup B_k} \implies F.$$

Applying this principle to  $(A_k, B_k) = (E_k \cap E'_k, E_k \setminus E'_k)$  and to  $(A_k, B_k) = (E_k \cap E'_k, E'_k \setminus E_k)$  proves (3.17).

Finally, appeal to Theorem 3.1 to see that (3.17) is equivalent to (3.16).  $\square$

#### 4. PROOF OF THEOREMS 3.1 AND 3.2, AND OF THEOREM 2.2

Throughout this section, assume (as in the theorems) that  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$ ,  $\mu(X) = \infty$ , pointwise dual ergodic with  $a_T \in \mathcal{R}_\alpha$  for some  $\alpha \in [0, 1]$ , and let  $b_T$  be asymptotically inverse to  $a_T$ . Suppose that  $Y$  is a DK-set, w.l.o.g. with  $\mu(Y) = 1$ , and that  $E_k \subseteq Y$ ,  $k \geq 1$ , are sets of positive finite measure with  $\mu(E_k) \rightarrow 0$ .

Our argument exploits the Ansatz of [PSZ2] (which goes back to [DE]), and uses several auxiliary facts already mentioned or established there. The following decomposition (Lemma 5.3 of [PSZ2]) is our starting point.

**Lemma 4.1 (Decomposing according to the last visit before time  $n$ ).** *Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$ , and  $A, B \in \mathcal{A}$ . Then*

$$(4.1) \quad \mu(A) = \mu(A \cap \{\varphi_B > n\}) + \sum_{l=1}^n \int_{B \cap \{\varphi_B > n-l\}} \widehat{T}^l 1_A d\mu \quad \text{for } n \geq 0.$$

As our goal is to prove (3.8) and (3.9), we define

$$(4.2) \quad R_k := \mu(E_k) a_T(\varphi_{E_k}), \quad \text{for } k \geq 1,$$

and denote the relevant distribution functions by  $\tilde{G}_k$  and  $G_k$ ,

$$(4.3) \quad \tilde{G}_k(t) := \mu_{E_k}(\mathbf{R}_k \leq t), \quad G_k(t) := \mu_Y(\mathbf{R}_k \leq t) \quad \text{for } k \geq 1, t \in [0, \infty).$$

It is convenient to set, for  $t \in [0, \infty)$  and  $k \geq 1$ ,

$$(4.4) \quad n_k^{[t]} := b_T(t/\mu(E_k)),$$

and, for  $l \in \{0, \dots, n_k^{[t]}\}$ ,  $\vartheta_{k,l}^{[t]} := \mu(E_k) \cdot a_T(n_k^{[t]} - l)$ . These allow us to represent the most important events as

$$(4.5) \quad \{\mathbf{R}_k > t\} = \{\varphi_{E_k} > n_k^{[t]}\}, \quad \text{while} \quad \{\mathbf{R}_k > \vartheta_{k,l}^{[t]}\} = \{\varphi_{E_k} > n_k^{[t]} - l\}.$$

Note that, for any fixed  $t$  and  $k$ ,  $l \mapsto \vartheta_{k,l}^{[t]}$  is non-increasing.

In case  $\alpha \in (0, 1]$ , given  $\rho \in [0, 1)$  and a positive sequence  $(l_k)_{k \geq 1}$ ,

$$(4.6) \quad \text{if } l_k \sim \rho \cdot n_k^{[t]}, \quad \text{then} \quad \vartheta_{k,l_k}^{[t]} \sim t \cdot (1 - \rho)^\alpha \quad \text{as } k \rightarrow \infty.$$

Moreover, by the DK-property of  $Y$ , if  $0 \leq c_1 < c_2$ , then

$$(4.7) \quad \sum_{j=c_1 n}^{c_2 n-1} \hat{T}^j 1_Y \sim (c_2^\alpha - c_1^\alpha) \cdot a_n \quad \text{as } n \rightarrow \infty, \text{ uniformly mod } \mu \text{ on } Y.$$

We are now ready for the

*Proof. Proof of Theorem 3.2, case  $\alpha \in (0, 1]$ .* (i) Assume that  $\tilde{G}_k \Rightarrow \tilde{G}$ , and define  $G$  via

$$(4.8) \quad G(t) = \alpha t \int_0^1 [1 - \tilde{G}(t(1-r)^\alpha)] r^{\alpha-1} dr \quad \text{for } t \in [0, \infty).$$

We are going to prove that there is a dense subset  $\mathcal{T}$  of  $[0, \infty)$  such that

$$(4.9) \quad G_k(t) \longrightarrow G(t) \quad \text{for } t \in \mathcal{T},$$

It is then immediate that  $G$  is a sub-probability distribution function on  $[0, \infty)$  for which  $G_k \Rightarrow G$ .

To this end, let  $\mathcal{T}$  be the set of those continuity points  $t \in (0, \infty)$  of  $\tilde{G}$  with the property that for all integers  $0 \leq m \leq M$ , the  $t(1 - \frac{m}{M})^\alpha$  also are continuity points of  $\tilde{G}$ . The complement of this set is only countable.

Henceforth, we fix some  $t \in \mathcal{T}$ , and abbreviate  $n_k := n_k^{[t]}$  and  $\vartheta_{k,i} := \vartheta_{k,i}^{[t]}$ .

Lemma 4.1, with  $A := Y$  and  $B := E_k$ , and  $n := n_k$ , gives

$$(4.10) \quad G_k(t) = 1 - \mu(Y \cap \{\varphi_{E_k} > n_k\}) = \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \hat{T}^l 1_Y d\mu.$$

We are going to prove, for  $k \rightarrow \infty$ , that

$$(4.11) \quad \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \hat{T}^l 1_Y d\mu \longrightarrow \alpha t \int_0^1 [1 - \tilde{G}(t(1-r)^\alpha)] r^{\alpha-1} dr.$$

(ii) Since  $b_T \in \mathcal{R}_{1/\alpha}$ , we have  $n_k \sim t^{1/\alpha} b_T(1/\mu(E_k))$  as  $k \rightarrow \infty$ . Fix some  $M \geq 1$ , and take any  $\varepsilon \in (0, 1)$ . Decomposing the sum in (4.11) into  $M$  sections and

recalling (4.5), we find that

$$\begin{aligned} \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l 1_Y d\mu &= \sum_{m=0}^{M-1} \sum_{l=\lfloor \frac{m}{M} n_k \rfloor + 1}^{\lfloor \frac{m+1}{M} n_k \rfloor} \int_{E_k \cap \{\mathbf{R}_k > \vartheta_{k,l}\}} \widehat{T}^l 1_Y d\mu \\ &\leq \sum_{m=0}^{M-1} \int_{E_k \cap \{\mathbf{R}_k > \vartheta_{k, \lfloor \frac{m}{M} n_k \rfloor + 1}\}} \sum_{l=\lfloor \frac{m}{M} n_k \rfloor + 1}^{\lfloor \frac{m+1}{M} n_k \rfloor} \widehat{T}^l 1_Y d\mu, \end{aligned}$$

where the second step uses that, by monotonicity of  $l \mapsto \vartheta_{k,l}$ ,  $\{\mathbf{R}_k > \vartheta_{k,l}\} \subseteq \{\mathbf{R}_k > \vartheta_{k, \lfloor (m+1)n_k/M \rfloor}\}$  for  $l \leq (m+1)n_k/M$ . In view of (4.7) and  $a_T(n_k) = t/\mu(E_k)$  we have, for  $m \geq 0$ ,

$$\sum_{l=\lfloor \frac{m}{M} n_k \rfloor + 1}^{\lfloor \frac{m+1}{M} n_k \rfloor} \widehat{T}^l 1_Y \sim \left( \left( \frac{m+1}{M} \right)^\alpha - \left( \frac{m}{M} \right)^\alpha \right) \frac{t}{\mu(E_k)} \quad \text{as } k \rightarrow \infty, \\ \text{uniformly mod } \mu \text{ on } Y.$$

Since  $(\frac{m+1}{M})^\alpha - (\frac{m}{M})^\alpha \leq \alpha \frac{1}{M} (\frac{m}{M})^{\alpha-1}$  by the mean-value theorem, we thus get

$$\sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l 1_Y d\mu \leq e^\varepsilon \alpha t \sum_{m=1}^{M-1} \mu_{E_k} \left( \mathbf{R}_k > \vartheta_{k, \lfloor \frac{m}{M} n_k \rfloor} \right) \left( \frac{m}{M} \right)^{\alpha-1} \frac{1}{M}$$

for  $k \geq K = K(M, \varepsilon)$ . By our choice of  $t$ ,  $\tilde{G}$  is continuous at each  $t(1 - \frac{m+1}{M})^\alpha$ , so that (4.6) ensures, as  $k \rightarrow \infty$ ,

$$\begin{aligned} \mu_{E_k} \left( \mathbf{R}_k > \vartheta_{k, \lfloor \frac{m}{M} n_k \rfloor} \right) &= 1 - \tilde{G}_k \left( \vartheta_{k, \lfloor \frac{m}{M} n_k \rfloor} \right) \\ &\longrightarrow 1 - \tilde{G} \left( t \left( 1 - \frac{m+1}{M} \right)^\alpha \right). \end{aligned}$$

Combining this with the above, and letting  $\varepsilon \searrow 0$ , we obtain

$$\overline{\lim}_{k \rightarrow \infty} \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^j 1_Y d\mu \leq \alpha t \sum_{m=1}^{M-1} \left[ 1 - \tilde{G} \left( t \left( 1 - \frac{m+1}{M} \right)^\alpha \right) \right] \left( \frac{m}{M} \right)^{\alpha-1} \frac{1}{M}.$$

Now  $M \rightarrow \infty$  yields

$$\overline{\lim}_{k \rightarrow \infty} \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^j 1_Y d\mu \leq \alpha t \int_0^1 [1 - \tilde{G}(t(1-r)^\alpha)] r^{\alpha-1} dr.$$

A parallel argument proves the corresponding lower estimate,

$$\underline{\lim}_{k \rightarrow \infty} \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^j 1_Y d\mu \geq \alpha t \int_0^1 [1 - \tilde{G}(t(1-r)^\alpha)] r^{\alpha-1} dr,$$

and hence our claim (4.11).

(iii) Now assume that  $G_k \Rightarrow G$ . Our goal is to show that  $\tilde{G}_k \Rightarrow \tilde{G}$  with  $\tilde{G}$  satisfying (4.8). In view of the Helly selection theorem we need only check that whenever  $\tilde{G}_{k_i} \Rightarrow \tilde{G}_*$  for some subsequence  $k_i \nearrow \infty$  of indices, this limit point  $\tilde{G}_*$  is indeed the unique sub-distribution function satisfying (4.8).

However, if we apply the conclusion of step (i) above, we see that  $\tilde{G}_{k_i} \Rightarrow \tilde{G}_*$  entails  $G_{k_i} \Rightarrow G_*$  with the pair  $(G_*, \tilde{G}_*)$  satisfying the desired integral equation.

Since  $G_k \Rightarrow G$  it is clear that  $G_* = G$ , so that in fact  $(G, \tilde{G}_*)$  satisfying the integral equation.  $\square$

A slight modification of the argument gives the

*Proof. Proof of Theorem 3.2, case  $\alpha = 0$ .* (i) Assume that  $\tilde{G}_k \Rightarrow \tilde{G}$ . By a subsequence-in-subsequence argument, we may assume that also  $G_k \Rightarrow G$  for some  $G \in \mathcal{F}$ . We are going to prove that for every point  $t > 0$  (henceforth fixed) at which both  $G$  and  $\tilde{G}$  are continuous,

$$(4.12) \quad G_k(t) \longrightarrow t[1 - \tilde{G}(t)] \quad \text{as } k \rightarrow \infty,$$

whence  $G(s) = s[1 - \tilde{G}(s)]$ ,  $s > 0$ . Abbreviate  $n_k := n_k^{[t]}$ . Just as in the  $\alpha \in (0, 1]$  case,  $G_k(t)$  is given by (4.10).

Observe that  $\{\varphi_{E_k} > n_k/2\} = \{\mathbf{R}_k > \mu(E_k) a(n_k/2)\}$ , where, due to slow variation of  $a_T$ ,  $\mu(E_k) a(n_k/2) \sim \mu(E_k) a(n_k) = t$  as  $k \rightarrow \infty$ . Since  $\tilde{G}$  is continuous at  $t$ , we thus see that

$$(4.13) \quad \begin{aligned} \mu_{E_k}(\varphi_{E_k} > n_k/2) &= 1 - \tilde{G}_k(\mu(E_k) a(n_k/2)) \\ &\longrightarrow 1 - \tilde{G}(t) \quad \text{as } k \rightarrow \infty. \end{aligned}$$

(ii) By the DK-property and  $a(n_k/2) \sim a(n_k)$  we have

$$\sum_{l=n_k/2+1}^{n_k} \hat{T}^l 1_Y = o(a(n_k)) \quad \text{as } k \rightarrow \infty, \text{ uniformly mod } \mu \text{ on } Y,$$

and hence, since  $a(n_k) = t/\mu(E_k)$  and  $E_k \subseteq Y$ , we find that

$$(4.14) \quad \begin{aligned} \sum_{l=n_k/2+1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \hat{T}^l 1_Y d\mu &\leq \int_{E_k} \sum_{l=n_k/2+1}^{n_k} \hat{T}^l 1_Y d\mu \\ &= \mu(E_k) o(a(n_k)) \\ &\longrightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

On the other hand, since  $\{\varphi_{E_k} > n_k - l\} \subseteq \{\varphi_{E_k} > n_k/2\}$  for  $l \leq n_k/2$ , we can appeal to the DK-property and to (4.13) to conclude that

$$(4.15) \quad \begin{aligned} \sum_{l=1}^{n_k/2} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \hat{T}^l 1_Y d\mu &\leq \int_{E_k \cap \{\varphi_{E_k} > n_k/2\}} \sum_{l=1}^{n_k/2} \hat{T}^l 1_Y d\mu \\ &\sim a(n_k/2) \mu(E_k \cap \{\varphi_{E_k} > n_k/2\}) \\ &\sim a(n_k) \mu(E_k \cap \{\varphi_{E_k} > n_k/2\}) \\ &\sim t \mu_{E_k}(\varphi_{E_k} > n_k/2) \\ &\longrightarrow t[1 - \tilde{G}(t)] \quad \text{as } k \rightarrow \infty. \end{aligned}$$

In view of (4.10), (4.14) and (4.15) together give

$$\overline{\lim}_{k \rightarrow \infty} G_k(t) \leq t[1 - \tilde{G}(t)].$$

(iii) To also prove  $\lim_{k \rightarrow \infty} G_k(t) \geq t[1 - \tilde{G}(t)]$ , and hence (4.12), we need only observe that by arguments similar to the above,

$$\begin{aligned}
 G_k(t) &= \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \hat{T}^l 1_Y d\mu \geq \int_{E_k \cap \{\varphi_{E_k} > n_k\}} \sum_{l=1}^{n_k} \hat{T}^l 1_Y d\mu \\
 &\sim a(n_k) \mu(E_k \cap \{\varphi_{E_k} > n_k\}) \\
 &\sim t \mu_{E_k}(\varphi_{E_k} > n_k) \\
 (4.16) \quad &\longrightarrow t[1 - \tilde{G}(t)] \quad \text{as } k \rightarrow \infty.
 \end{aligned}$$

(iv) Conversely, if we start from the assumption that  $G_k \Rightarrow G$ , and want to show that  $\tilde{G}_k \Rightarrow \tilde{G}$  with  $\tilde{G}$  satisfying  $G(s) = s[1 - \tilde{G}(s)]$ ,  $s > 0$ , we can use the above by arguing as in the last step of the proof for the  $\alpha \in (0, 1]$  case.

(v) Finally, we check that  $G(t') - G(t) \leq t' - t$  for  $t' > t > 0$ . Let  $n'_k := n_k^{[t']}$  and use (4.10) twice to see that

$$G_k(t') \leq G_k(t) + \int_{E_k} \sum_{l=n_k+1}^{n'_k} \hat{T}^l 1_Y d\mu.$$

But

$$\int_{E_k} \sum_{l=n_k+1}^{n'_k} \hat{T}^l 1_Y d\mu \sim \mu(E_k) (a_T(n'_k) - a_T(n_k)) = t' - t \quad \text{as } k \rightarrow \infty,$$

and the desired estimate follows. This proves continuity of  $G$ .  $\square$

Now recall the following folklore principle (see e.g. Lemma 1 of [BZ]).

**Lemma 4.2 (Regular variation preserves distributional convergence).** *Assume that  $R_n$  and  $R$  are random variables taking values in  $(0, \infty)$ , and that  $\rho_n^{-1} R_n \Rightarrow R$  for normalizing constants  $\rho_n \rightarrow \infty$ . If  $B$  is regularly varying of index  $\beta \neq 0$ , then*

$$(4.17) \quad \frac{B(R_n)}{B(\rho_n)} \Rightarrow R^\beta.$$

This easily leads to

**Proof of Theorem 3.1.** Fix  $\alpha \in (0, 1]$ . Applying Lemma 4.2 proves that (3.4) is equivalent to (3.8) with  $\tilde{G}(t) = \tilde{F}(t^{1/\alpha})$ , while (3.5) is equivalent to (3.9) with  $G(t) = F(t^{1/\alpha})$ . This relation between  $(\tilde{G}, G)$  and  $(\tilde{F}, F)$  turns (3.10) into

$$F(t) = t^\alpha \int_0^1 [1 - \tilde{F}(t(1-r))] \alpha r^{\alpha-1} dr$$

which, after an obvious change of variables, becomes (3.6).  $\square$

Finally, we can establish the main positive result of Section 2.

**Proof of Theorem 2.2.** Assume otherwise, then there is some  $\delta > 0$  and a sequence  $(E_k)$  in  $Y \cap \mathcal{A}$ ,  $\mu(E_k) > 0$  such that  $\mu_{E_k}(\gamma_T(\mu(E_k)) \cdot \varphi_{E_k} > k) > \delta$  for  $k \geq 1$ . In view of Proposition 2.1 b), this sequence must satisfy  $\mu(E_k) \rightarrow 0$ .

Now Helly's selection theorem guarantees  $k_j \nearrow \infty$  such that

$$\mu_{E_{k_j}}(\gamma_T(\mu(E_{k_j})) \varphi_{E_{k_j}} \leq t) \Rightarrow \tilde{F}(t),$$



for some  $\tilde{F} \in \mathcal{F}$ . By our choice of  $(E_k)$ , the latter satisfies  $\sup_{t \in [0, \infty)} \tilde{F}(t) < 1 - \delta$ . But this contradicts Remark 3.1 c).  $\square$

## 5. PROVING CONVERGENCE TO $\mathcal{H}_\alpha$

**Characterizing convergence to  $\mathcal{H}_\alpha$ .** In the case of finite measure preserving systems, Theorem HLV is not only of interest in its own right, but it is also the basis of a method for proving convergence to the exponential distribution (see [HSV]). Indeed, it is easy to see that the only  $F \in \mathcal{F}$  which satisfies

$$F(t) = \int_0^t [1 - F(s)] ds \quad \text{for } t \geq 0,$$

is  $F = H_1$ , where  $H_1(t) := 1 - e^{-t}$ ,  $t > 0$ . The most prominent limit law is thus characterized as the unique distribution which can appear both as return- and as hitting-time limit for the same sequence of sets. In view of this and the Helly selection principle, one can prove convergence to  $\mathcal{E}$  of both return- and hitting-time distributions by showing that the two types of distributions are asymptotically the same.

Here, we obtain (with hardly any effort) a result which allows a parallel approach to proving convergence to  $\mathcal{H}_\alpha$  in systems with DK-sets and  $a_T \in \mathcal{R}_\alpha$ ,  $\alpha \in (0, 1]$ .

**Theorem 5.1 (Convergence to  $\mathcal{H}_\alpha$ ).** *Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$ ,  $\mu(X) = \infty$ . Assume that  $Y \in \mathcal{A}$  is a DK-set and that  $a_T \in \mathcal{R}_\alpha$  for some  $\alpha \in (0, 1]$ .*

*Suppose that  $E_k \subseteq Y$ ,  $k \geq 1$ , are sets of positive measure with  $\mu(E_k) \rightarrow 0$ . Then the normalized return-time distributions of the  $E_k$  converge to  $\mathcal{H}_\alpha$ ,*

$$(5.1) \quad \mu_{E_k}(\gamma_T(\mu(E_k)) \cdot \varphi_{E_k} \leq t) \implies H_\alpha(t) \quad \text{as } k \rightarrow \infty,$$

*if and only if the normalized hitting-time distributions converge to  $\mathcal{H}_\alpha$ ,*

$$(5.2) \quad \mu_Y(\gamma_T(\mu(E_k)) \cdot \varphi_{E_k} \leq t) \implies H_\alpha(t) \quad \text{as } k \rightarrow \infty,$$

*if and only if for a dense set of points  $t$  in  $(0, \infty)$ ,*

$$(5.3) \quad \mu_{E_k}(\gamma_T(\mu(E_k)) \varphi_{E_k} \leq t) - \mu_Y(\gamma_T(\mu(E_k)) \varphi_{E_k} \leq t) \longrightarrow 0 \quad \text{as } k \rightarrow \infty.$$

*Proof.* By Theorem 3.1 it is clear that (5.1) is equivalent to (5.2). Trivially, either of these statements therefore implies (5.3).

To prove the converse, start from (5.3), and assume for a contradiction that, say, (5.1) fails, so that by Helly's selection principle there is a subsequence  $k_j \nearrow \infty$  of indices and some  $\tilde{F} \in \mathcal{F}$ ,  $\tilde{F} \neq H_\alpha$ , such that (3.4) holds along that subsequence. By Theorem 3.1, so does (3.5), where  $F$  and  $\tilde{F}$  are related by (3.6). But (5.3) ensures that  $\tilde{F} = F$ , which in view of Remark 3.1 f) contradicts  $\tilde{F} \neq H_\alpha$ .

Exactly the same argument works if we assume that (5.2) fails.  $\square$

**Sufficient conditions for convergence to  $\mathcal{H}_\alpha$ .** It is natural to review the abstract distributional limit theorem of [PSZ2], which gives sufficient conditions for convergence to  $\mathcal{H}_\alpha$ , in the light of the preceding result. Below we restate Theorem 4.1 of [PSZ2]. The conditions of that result are somewhat technical in order to cover sequences of cylinders around typical points of the concrete systems studied there.

Roughly speaking, the meaning of the conditions is this: In view of Theorem 5.1 we have to compare the distributions of  $\gamma_T(\mu(E_k)) \cdot \varphi_{E_k}$  with respect to the two measures  $\mu_{E_k}$  and  $\mu_Y$ , with densities  $\mu(E_k)^{-1} 1_{E_k}$  and  $1_Y$ , respectively. If  $Y$  is a  $\mathcal{U}$ -uniform set with  $1_Y \in \mathcal{U}$ , then any  $u \in \mathcal{U}$  will be as good as  $1_Y$ , and we ensure that after a (point-dependent) number of steps, the operator  $\hat{T}$  turns  $\mu(E_k)^{-1} 1_{E_k}$  into an element of  $\mathcal{U}$ . First, there are  $z_k$  steps during which the images of  $E_k$  (the support of the push-forward densities) grow to a definite size. Here  $z_k$  should not be too large, and we don't want many points to return to  $E_k$  during this phase (conditions (5.4) and (5.5)). After these  $z_k$  steps we see a moderately nice density  $w_k$ , which will be chopped up into parts  $\pi_{k,\iota} w_{k,\iota}$ ,  $\iota \geq 0$ , which need  $\iota$  further steps until they end up in  $\mathcal{U}$ . The distribution of the position-dependent second delay  $\iota$  is captured by the variable  $\Upsilon_k$  which should be small compared to  $\gamma_T(\mu(E_k)) \cdot \varphi_{E_k}$  (condition (5.7) below).

**Theorem 5.2 (Convergence to  $\mathcal{H}_\alpha$  via efficient regeneration, [PSZ2]).** *Let  $T$  be a c.e.m.p.t. on  $(X, \mathcal{A}, \mu)$ ,  $\mu(X) = \infty$ , pointwise dual ergodic with  $a_T \in \mathcal{R}_\alpha$  for some  $\alpha \in (0, 1]$ . Suppose that  $Y$  is a  $\mathcal{U}$ -uniform set with  $1_Y \in \mathcal{U} \subseteq \mathcal{D}(\mu)$ , and that  $E_k \subseteq Y$ ,  $k \geq 1$ , are sets of positive measure with  $\mu(E_k) \rightarrow 0$ , and that  $z_k \geq 0$  are integers such that*

$$(5.4) \quad z_k \cdot \mu(E_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and

$$(5.5) \quad \mu_{E_k}(\varphi_{E_k} \leq z_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Assume, in addition, that  $w_k := \hat{T}^{z_k}(1_{E_k}/\mu(E_k)) = \sum_{\iota \geq 0} \pi_{k,\iota} w_{k,\iota}$  with densities  $w_{k,\iota} \in \mathcal{D}(\mu)$  satisfying

$$(5.6) \quad 1_Y \hat{T}^j w_{k,\iota} = 0 \text{ for } 1 \leq j < \iota, \quad \text{while} \quad \hat{T}^\iota w_{k,\iota} \in \mathcal{U},$$

and weights  $\pi_{k,\iota} \geq 0$  such that any random variables  $\Upsilon_k$  with  $\Pr[\Upsilon_k = \iota] = \pi_{k,\iota}$  satisfy

$$(5.7) \quad \gamma_T(\mu(E_k)) \cdot \Upsilon_k \xrightarrow{\Pr} 0 \quad \text{as } k \rightarrow \infty.$$

Then the return-time distributions of the  $E_k$  converge to  $\mathcal{H}_\alpha$ ,

$$(5.8) \quad \mu_{E_k}(\gamma_T(\mu(E_k)) \cdot \varphi_{E_k} \leq t) \rightarrow \Pr[\mathcal{E}^{\frac{1}{\alpha}} \mathcal{G}_\alpha \leq t] \quad \text{as } k \rightarrow \infty,$$

and so do the hitting-time distributions,

$$(5.9) \quad \mu_Y(\gamma_T(\mu(E_k)) \cdot \varphi_{E_k} \leq t) \rightarrow \Pr[\mathcal{E}^{\frac{1}{\alpha}} \mathcal{G}_\alpha \leq t] \quad \text{as } k \rightarrow \infty.$$

We indicate how this can be derived from Theorem 5.1.

**Reorganized proof of Theorem 5.2.** (i) Assume w.l.o.g. that  $\mu(Y) = 1$ . Using the notations introduced above, we are going to prove that for arbitrary  $t \in (0, \infty)$  (henceforth fixed) and  $\varepsilon \in (0, \frac{2}{3} \log 3)$ ,

$$(5.10) \quad \tilde{G}_k(t) < e^\varepsilon G_k(t) + \varepsilon \quad \text{for } k \geq K_0(\varepsilon),$$

which is the ‘‘upper half’’ of (5.3). The ‘‘lower half’’ can be established by a similar argument, the details of which we omit.

Choose  $M \geq 1$  so large that  $[(\frac{M+1}{M})^\alpha - 1]t < \varepsilon/4$ . By the DK-property of  $Y$ , there is some  $K'(\varepsilon)$  such that for  $k \geq K'(\varepsilon)$  and  $0 \leq m \leq M$ ,

$$(5.11) \quad \sum_{l=\lfloor \frac{m}{M}n_k \rfloor + 1}^{\lfloor \frac{m+1}{M}n_k \rfloor} \hat{T}^l(1_Y) = e^{\pm \varepsilon/2} \left[ \left( \frac{m+1}{M} \right)^\alpha - \left( \frac{m}{M} \right)^\alpha \right] a_T(n_k) \quad \text{on } Y.$$

Here,  $s = e^{\pm \delta} s'$  is short for  $e^{-\delta} s' \leq s \leq e^{\delta} s'$ . In particular, since  $a_T(n_k)\mu(E_k) = t$ , we see that for  $k \geq K'(\varepsilon)$ ,

$$(5.12) \quad \int_{E_k} \sum_{l=n_k+1}^{\lfloor \frac{M+1}{M}n_k \rfloor} \hat{T}^l(1_Y) d\mu < e^{\varepsilon/2} \left[ \left( \frac{M+1}{M} \right)^\alpha - 1 \right] a_T(n_k)\mu(E_k) < \frac{\varepsilon e^{\varepsilon/2}}{4}.$$

On the other hand we can use Lemma 5.2 of [PSZ2] (with  $\bar{n}_k \sim n_k - \frac{M}{m}z_k \sim n_k$ ) to also ensure that for  $k \geq K'(\varepsilon)$  and  $0 \leq m \leq M$ ,

$$(5.13) \quad \sum_{j=\lfloor \frac{m}{M}n_k \rfloor + 1 - z_k}^{\lfloor \frac{m+1}{M}n_k \rfloor - z_k} \hat{T}^j(w_k) < e^{\varepsilon/2} \left[ \left( \frac{m+1}{M} \right)^\alpha - \left( \frac{m}{M} \right)^\alpha \right] a_T(n_k) \quad \text{on } Y.$$

(ii) According to Lemma 4.1 (with  $A := B := E_k$ ) we have

$$(5.14) \quad \tilde{G}_k(t) = \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \hat{T}^l(\mu(E_k)^{-1} 1_{E_k}) d\mu.$$

The contribution of the first  $z_k$  terms is asymptotically negligible, as (5.5) ensures

$$(5.15) \quad \sum_{l=1}^{z_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \hat{T}^l 1_{E_k} d\mu_{E_k} \leq \sum_{l=1}^{z_k} \int_{E_k \cap \{\varphi_{E_k} > z_k - l\}} \hat{T}^l 1_{E_k} d\mu_{E_k} \\ = \mu_{E_k}(\varphi_{E_k} \leq z_k) < \varepsilon/4$$

for  $k \geq K''(\varepsilon)$ . Now estimate the main part of the sum in (5.14),

$$\begin{aligned} & \sum_{m=0}^{M-1} \sum_{l=(\lfloor \frac{m}{M}n_k \rfloor + 1) \vee z_k}^{\lfloor \frac{m+1}{M}n_k \rfloor} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \hat{T}^l(\mu(E_k)^{-1} 1_{E_k}) d\mu \\ &= \sum_{m=0}^{M-1} \sum_{l=(\lfloor \frac{m}{M}n_k \rfloor + 1) \vee z_k}^{\lfloor \frac{m+1}{M}n_k \rfloor} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \hat{T}^{l-z_k}(w_k) d\mu \\ &\leq \sum_{m=0}^{M-1} \int_{E_k \cap \{\varphi_{E_k} > n_k - (m+1)n_k/M\}} \sum_{j=(\lfloor \frac{m}{M}n_k \rfloor + 1 - z_k) \vee 0}^{\lfloor \frac{m+1}{M}n_k \rfloor - z_k} \hat{T}^j(w_k) d\mu \quad \dots \end{aligned}$$

which uses  $E_k \cap \{\varphi_{E_k} > n_k - l\} \subseteq E_k \cap \{\varphi_{E_k} > n_k - (m+1)n_k/M\}$  for  $\lfloor \frac{m}{M}n_k \rfloor < l \leq \lfloor \frac{m+1}{M}n_k \rfloor$ . Due to (5.13) and (5.11), we can continue to estimate this expression,

$$\begin{aligned}
& \dots < e^{\varepsilon/2} \sum_{m=0}^{M-1} \int_{E_k \cap \{\varphi_{E_k} > n_k - (m+1)n_k/M\}} \left[ \left( \frac{m+1}{M} \right)^\alpha - \left( \frac{m}{M} \right)^\alpha \right] a_T(n_k) d\mu \\
& < e^\varepsilon \sum_{m=0}^{M-1} \int_{E_k \cap \{\varphi_{E_k} > n_k - (m+1)n_k/M\}} \sum_{l=\lfloor \frac{m}{M} n_k \rfloor + 1}^{\lfloor \frac{m+1}{M} n_k \rfloor} \widehat{T}^l(1_Y) d\mu \\
& \leq e^\varepsilon \sum_{m=0}^{M-1} \sum_{l=\lfloor \frac{m}{M} n_k \rfloor + 1}^{\lfloor \frac{m+1}{M} n_k \rfloor} \int_{E_k \cap \{\varphi_{E_k} > n_k - l - n_k/M\}} \widehat{T}^l(1_Y) d\mu \\
& \leq e^\varepsilon \sum_{m=1}^M \sum_{l=\lfloor \frac{m}{M} n_k \rfloor + 1}^{\lfloor \frac{m+1}{M} n_k \rfloor} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l(1_Y) d\mu \\
& \leq e^\varepsilon \left[ \sum_{m=0}^{M-1} \sum_{l=\lfloor \frac{m}{M} n_k \rfloor + 1}^{\lfloor \frac{m+1}{M} n_k \rfloor} + \sum_{l=n_k+1}^{\lfloor \frac{M+1}{M} n_k \rfloor} \right] \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l(1_Y) d\mu.
\end{aligned}$$

In view of (5.14), this together with (5.15) and (5.12) gives

$$\widetilde{G}_k(t) < e^\varepsilon G_k(t) + (1 + e^{3\varepsilon/2})\varepsilon/4 \quad \text{for } k \geq K_0(\varepsilon)$$

with  $K_0(\varepsilon) := K'(\varepsilon) \vee K''(\varepsilon)$ , because  $G_k(t) = \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l(1_Y) d\mu$  (by Lemma 4.1 with  $A := Y$  and  $B := E_k$ ). However,  $(1 + e^{3\varepsilon/2})/4 < 1$ , proving (5.10).  $\square$

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